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The inverted harmonic oscillator: some statistical properties

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Abstract. We consider the derivation of some statistical results for the 'inverted' harmonic oscillator (one with negative kinetic and potential energies), which are equivalent to ones that exist for the more familiar simple harmonic oscillator. It is an object that is frequently employed in modelling amplifiers in quantum optics, but also arises in statistical and quantum mechanics and is of interest in its own right. We hope that this account may help to elucidate its role as a potential amplifier.

1. Introduction

There has been much recent interest in optical amplification. Early hopes of cloning photons by stimulated emission were confounded when it was shown that the amplifier instead acts as a source of noise, statistically independent of the input field [1, 2]. It was shown that both linear amplification and linear damping in a quantum system coexist with external noise fluctuations (see [3, 4] and references therein). Both coherence and correlation properties of the output light have been researched [5-7]. There have also been studies of minimum limits for the noise that accompanies amplification [8], as well as many derivations [9, 10] of limits for squeezing-preserving intensity gains of squeezed output from linear light amplifiers, with subsequent attempts [11-15] to overcome the latter bounds for atomic amplifiers. In particular, inverted oscillator light amplifiers have been the subject of much investigation, where such objects have been used to discuss fundamental problems in the quantum theory of measurement [16], the short-time behaviour in superfluorescence [17], and of a spin magnetic moment in a magnetic field [18].

Below, we briefly outline some properties of the inverted harmonic oscillator which are analogues of important relations obeyed by the familiar 'upright' harmonic oscillator, in terms of which modes of the electromagnetic field are described. We adopt Glauber's model of an inverted oscillator [16] for the quantum amplifier, which assumes the following Hamiltonian:

$$H = -\tfrac{1}{2}(p^2 + \omega^2 q^2) = -\hbar\omega(c^\dagger c + \tfrac{1}{2})$$

so that the annihilation operator evolves as $c(t) = c(0)\,\mathrm{e}^{i\omega t}$.

The nth stationary state $|n\rangle$ obeys

$$|n\rangle = (c^{\dagger n}/\sqrt{n!})|0\rangle \qquad c|n\rangle = \sqrt{n}|n-1\rangle$$
$$c^\dagger c|n\rangle = n|n\rangle \qquad E_n = -\hbar\omega(n + \tfrac{1}{2}).$$

We can obtain our inverted oscillator from the usual simple harmonic oscillator, described by operators a, $a^\dagger$, by the substitutions $\omega \to -\omega$ and $a \to c^\dagger$, as in figure 1.

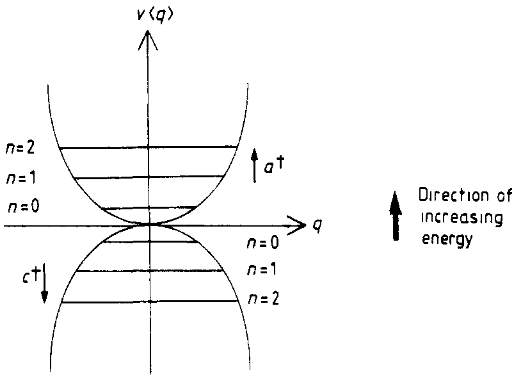

Figure 1. Harmonic and inverted oscillator potentials and stationary states.

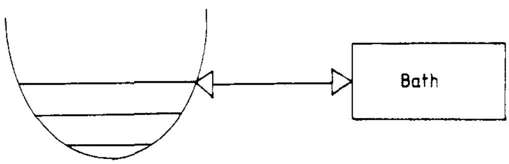

Figure 2. Harmonic oscillator coupled to a heat bath of oscillators.

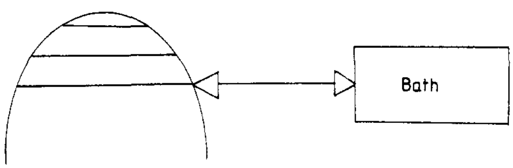

Figure 3. Inverted oscillator coupled to a heat bath of oscillators.

Important differences between the models of figures 2 and 3 have been found; see for example [16] and references therein. Physically, figure 3 represents a collection of atoms (the inverted oscillator), losing their energy to a bath of cavity field modes; the atoms may have all their 'input' modes occupied, but only one mode is amplified.

Perhaps the most important message here is that, for linear systems, dissipation and amplification may be regarded, in a sense, as different manifestations of the same phenomenon of response to within the approximations that we have employed. This is independently borne out by the research of Dupertuis *et al* [13–15].

2. The fluctuation amplification theorem

Following Louisell's derivation [19] of the fluctuation dissipation theorem, for the model of figure 2, we obtain a connection between amplification and quantum (vacuum) fluctuations for an inverted oscillator coupled to a heat bath of oscillators (see figure 3). Let c and $\{b_j\}$ be the annihilation operators for the inverted oscillator and bath oscillator modes respectively. The system is described by the Hamiltonian

$$H_{\mathrm{T}} = -\hbar\omega_c(c^+c+\tfrac{1}{2}) + \hbar\sum_j(\omega_j b_j^+ b_j + \tfrac{1}{2}) + H_{\mathrm{I}} \tag{1a}$$

where

$$H_1 = \hbar \sum_j (\kappa_j b_j c + \kappa_j^* c^\dagger b_j^\dagger) \tag{1b}$$

and $\{\kappa_j\}$ are a set of coupling constants. Henceforth we omit the zero-point energies, and set $\hbar = 1$.

Working in the Heisenberg picture, we find the following equations of motion:

$$\dot{b}_j(t) = -\mathrm{i}\omega_j b_j - \mathrm{i}\kappa_j^* c^\dagger \tag{2a}$$

$$\dot{d}(t) = \sum_j |\kappa_j|^2 \int_0^t d(t') \exp[\mathrm{i}(\omega_j - \omega_c)(t - t')]\,\mathrm{d}t' + G_d \tag{2b}$$

where d is the interaction picture operator, given by $d(t) = c(t)\exp(-\mathrm{i}\omega_c t)$ and $G_d = -\mathrm{i}\sum_j \kappa_j^* b_j^\dagger(0)\exp[\mathrm{i}(\omega_j - \omega_c)t]$ is the Langevin (random) force giving rise to fluctuations in the system. We adopt the Weisskopf-Wigner approximation to produce a truly dissipative subsystem, the reservoir. Thus we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t} d(t) = \frac{\gamma}{2} d(t) + G_d(t). \tag{3}$$

The term $\frac{1}{2}\gamma d(t)$ signifies the mean drift motion, whilst $G_d(t)$ is a noise source. Assuming the system to be Markovian, we find, on tracing over the reservoir in thermal equilibrium, that

$$\langle d(t)\rangle_{\mathrm{R}} = \exp(\gamma t/2) c(0). \tag{4}$$

We can thus justify calling $\gamma = 2\pi g(\omega_c)|\kappa(\omega_c)|^2$ the 'amplification constant'. Here, $g(\omega_c)$ is the density of states within the bath at the central oscillator frequency. After some standard manipulation [19], which involves evaluating the cross-correlation function $F_{d^\dagger d}(t_1 - t_2) \equiv \langle G_{d^\dagger}(t_1) G_d(t_2)\rangle_{\mathrm{R}}$, by integration over ω_j, assuming $g(\omega_j)|\kappa(\omega_j)|^2 \bar{n}(\omega_j)$ is a slowly varying function of ω_j, we integrate over τ to arrive at

$$\gamma = \frac{1}{\bar{n}} \int_{-\infty}^{\infty} \langle G_d(\tau) G_{d^\dagger}(0)\rangle_{\mathrm{R}}\,\mathrm{d}\tau \tag{5a}$$

where $\bar{n}(\omega_j) = \langle b_j^\dagger(0) b_j(0)\rangle_{\mathrm{R}}$ is the mean photon number in the jth bath oscillator and we have gone to the continuum limit in j, the bath variables. This takes the form of a 'fluctuation amplification theorem', wherein the system amplification rate γ and the reservoir fluctuating forces which introduce fluctuations into the system are interrelated. An alternative, but equally valid, expression for the theorem is

$$\gamma = \frac{1}{\bar{n}+1} \int_{-\infty}^{\infty} \langle G_{d^\dagger}(\tau) G_d(0)\rangle_{\mathrm{R}}\,\mathrm{d}\tau. \tag{5b}$$

3. The Einstein relation and spectra

In similar fashion, we can derive a Langevin equation for the photon number in our inverted oscillator (see also [4]). We find under the Markov approximation that

$$\frac{\mathrm{d}}{\mathrm{d}t} d^\dagger d = \gamma d^\dagger d + \gamma(\bar{n}+1) + G_{d^\dagger d} \tag{6}$$

where the relevant Langevin force is

$$G_{d^\dagger d}(t) = d^\dagger(t_c) G_d(t) + G_{d^\dagger}(t) d(t_c) \tag{7}$$

and t_c is a time such that $0<\tau_c \ll t-t_c \ll \gamma^{-1}$. The correlation time τ_c is a minimum time in which the Langevin forces change significantly. We define the 'diffusion coefficient' by

$$2\langle D_{d^+d}\rangle_R \equiv \frac{1}{\Delta t}\int_t^{t+\Delta t} dt_1 \int_t^{t+\Delta t} dt_2 \langle G_{d^+}(t_1) G_d(t_2)\rangle_R \tag{8}$$

with $\tau_c \ll \Delta t \ll \gamma^{-1}$. With the assumption that the integrand assumes its peak value when $\omega_j = \omega_c$, we can extend the lower limits to $-\infty$. Finally, we have that

$$2\langle D_{d^+d}\rangle_R = \frac{d}{dt}\langle d^+ d\rangle_R - \left\langle d^+\left[\frac{d}{dt}d(t) - G_d\right]\right\rangle_R - \left\langle\left[\frac{d}{dt}d^+(t) - G_{d^+}\right]d\right\rangle_R \tag{9}$$

which is the Einstein relation for the diffusion constant. For example, for our problem, using equations (3) and (6),

$$2\langle D_{d^+d}\rangle_R = \gamma(\bar{n}+1). \tag{10}$$

That is, the diffusion constant is given in terms of the drift terms, which determine the mean underlying motion of a system. We can generalise equation (9), so as to include spin-flip operators describing two-level atoms, or oscillators which obey Fermi–Dirac statistics, as in [20].

Next, we define the fluctuation, intensity and photon number spectra (unnormalised), for our example, respectively, by

$$I_1 = \int_{-\infty}^{\infty} e^{-i\omega t}\langle c^+(t) c(0)\rangle \, dt \tag{11a}$$

$$I_2 = \int_{-\infty}^{\infty} e^{-i\omega t}\langle c^+(0) c^+(t) c(t) c(0)\rangle \, dt \tag{11b}$$

$$I_3 = \int_{-\infty}^{\infty} e^{-i\omega t}\langle c^+(t) c(t) c^+(0) c(0)\rangle \, dt. \tag{11c}$$

Evaluating these using equations (3) and (6), we find that

$$I_1 = -\frac{\gamma\langle c^+(0)c(0)\rangle}{(\omega+\omega_c)^2 + (\gamma/2)^2} \tag{12a}$$

$$I_2 = -\frac{2\gamma}{\gamma^2+\omega^2}\{\langle c^{+2}(0)c^2(0)\rangle + (\bar{n}+1)\langle c^+(0)c(0)\rangle\} - 2\pi(\bar{n}+1)\langle c^+(0)c(0)\rangle\delta(\omega) \tag{12b}$$

$$I_3 = I_2 - \frac{2\gamma}{\gamma^2+\omega^2}\langle c^+(0)c(0)\rangle. \tag{12c}$$

$\bar{n}$ is the mean photon number in a bath oscillator at frequency ω_c. The positivity of the spectra is borne out, since c^+c represents a de-excitation, that is a negative energy, on the absolute energy scale. This arises because of the even powers of operators in their definitions (11), so positive definiteness is assured. For comparison, the corresponding expression for I_1 for the model of figure 2 is

$$I_1(a, a^+) = \frac{\gamma\langle a^+(0)a(0)\rangle}{(\omega-\omega_c)^2+(\gamma/2)^2}.$$

4. Discussion

Because we are dealing with a Markovian evolution, the quantum regression theorem (QRT) will be of relevance. In our case, we can use it to find an expression for γ. We need to evaluate $\langle G_d(\tau)G_{d^+}(0)\rangle_{\mathrm{R}}$, which arises in the expression for the fluctuation amplification rate, of equation (5a). Now $G_d(t) = -\mathrm{i}\Sigma_j \kappa_j^* b_j^\dagger(0)\exp[\mathrm{i}(\omega_j-\omega_c)t]$. Hence by the QRT

$$\langle G_d(t+\tau)\rangle = \sum_j \alpha_j(\tau)\langle G_d^{(j)}(t)\rangle \tag{13}$$

where $\alpha_j(\tau) = -\mathrm{i}\kappa_j^* \exp[\mathrm{i}(\omega_j-\omega_c)(t+\tau)]$ and $G_d^{(j)}(t) \equiv b_j^\dagger(0)$. Temporarily assuming γ to be time dependent, we find that

$$\begin{aligned}\gamma(t) &= \frac{1}{\bar{n}}\int_{-\infty}^{\infty} \langle G_d(t+\tau)G_d^\dagger(\tau)\rangle_{\mathrm{R}}\,\mathrm{d}\tau \\ &= \frac{1}{\bar{n}}\sum_j \left\langle b_j^\dagger(0)\left(\mathrm{i}\sum_l \kappa_l b_l(0)\exp[-\mathrm{i}(\omega_l-\omega_c)t]\right)\right\rangle_{\mathrm{R}} \\ &\quad\times\int_{-\infty}^{\infty} -\mathrm{i}\kappa_j^*\exp[\mathrm{i}(\omega_j-\omega_c)(t+\tau)]\,\mathrm{d}\tau.\end{aligned} \tag{14}$$

If the reservoir is in thermal equilibrium, then $\langle b_j^\dagger(0)b_1(0)\rangle_{\mathrm{R}} = \delta_{j1}\bar{n}_j$. Setting $t=0$, and taking $|\kappa(\omega_j)|^2\bar{n}(\omega_j)$ to be a slowly varying function of ω_j, the sum becomes an integral, giving

$$\gamma = 2\pi g(\omega_c)|\kappa(\omega_c)|^2. \tag{15}$$

This is the response rate familiar from time-dependent first-order perturbation Fermi golden rule theory. It is interesting to note that (14) takes the form of a 'Kubo equation' [21, 22] which gives the response of a system to an oscillatory perturbation. Further, Nyquist's theorem [21, 23-25] in statistical mechanics relates the generalised resistance (an irreversible process) to the fluctuations of the generalised forces in linear dissipative systems. Thus spontaneous (i.e. vacuum stimulated) transitions may be considered as arising from the random fluctuations of the electromagnetic field in the vacuum state (the dissipative system), acting as an excited atom [26].

We conclude this section by noting the difference in structure of the Langevin equations of motion for harmonic and inverted oscillators, respectively,

$$\frac{\mathrm{d}}{\mathrm{d}t}f(t) = -\tfrac{1}{2}\gamma f(t) + G_f(t)$$

$$G_f(t) = -\mathrm{i}\sum_j \kappa_j b_j(0)\exp[-\mathrm{i}(\omega_j-\omega_c)t]$$

and

$$\frac{\mathrm{d}}{\mathrm{d}t}d(t) = \tfrac{1}{2}\gamma d(t) + G_d(t)$$

$$G_d(t) = -\mathrm{i}\sum_j \kappa_j^* b_j^\dagger(0)\exp[\mathrm{i}(\omega_j-\omega_c)t]$$

where $f(t) = a(t)\,\mathrm{e}^{\mathrm{i}\omega t}$.

Nevertheless, the dissipation width

$$\gamma = \frac{1}{\bar{n}}\int_{-\infty}^{\infty} \langle G_{f^+}(\tau)G_f(0)\rangle_{\mathrm{R}}\,\mathrm{d}\tau$$

is identical with the fluctuation width of ($5a$), as we have shown above using the QRT, or as can be proved by direct substitution.

5. Conclusions

We have discussed the exponentially increasing energy loss of an inverted harmonic oscillator by its linear coupling to a collection of harmonic oscillators from a statistical viewpoint. It was found that, just as for the equivalent linear damping problem, the central oscillator lost its energy to the bath, which in turn returned part of that energy in a random fashion back to the primary oscillator.

The operation of a quantum linear amplifier adds Langevin noise to the emitted signal, thus degrading the efficiency of the output by decreasing the signal-to-noise ratio. In going from attenuator problems to the analogous amplifier problems with which we have dealt above, we have found that the replacements $\omega \to -\omega$ and $a \to c^{\dagger}$ in the formalism have the same effect in the results as the naive substitutions

$$-\gamma \to \gamma \qquad \text{and} \qquad \bar{n} \rightleftharpoons \bar{n}+1.$$

The first of these changes is perhaps to be expected; the factor of unity in the second is an intrinsic spontaneous emission term, essential for the elucidation of quantum amplification. The Langevin equation (6) for the photon number is a good example of how both input photons and zero-point photons initially present are amplified. Prudence is necessary in fashioning the time evolution of physical systems comprising the inverted oscillator; for large enough times, the evolution becomes non-linear and the model breaks down.

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